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## LETTER TO THE EDITOR

**Spurious phase in a model for traffic on a bridge**

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Online at [stacks.iop.org/JPhysA/38/L659](http://stacks.iop.org/JPhysA/38/L659)**Abstract**

We present high-precision Monte Carlo data for the phase diagram of a two-species driven diffusive system, reminiscent of traffic across a narrow bridge. Earlier studies reported two phases with broken symmetry; the existence of one of these has been the subject of some debate. We show that the disputed phase disappears for sufficiently large systems and/or sufficiently low bulk mobility.

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**1. Introduction**

When driven far from thermal equilibrium, interacting many-particle systems display a wide variety of unexpected behaviours [1, 2]. Generic long-range correlations, new phase transitions and universality classes, and anomalously fast coarsening are amongst the most striking phenomena. In contrast to equilibrium systems, our notions of energy-entropy competition fail so that, even in low spatial dimensions, a plethora of novel phases and phase transitions have been discovered [1–3]. In the absence of a fundamental theoretical framework for nonequilibrium systems, these transitions are usually identified with the help of Monte Carlo simulations, and subtle finite-size analyses are essential in order to establish their nature in the thermodynamic limit. Only in a few cases can such questions be settled by recourse to exact solutions.

Asymmetric exclusion processes (ASEPs) constitute a particularly simple class of far-from-equilibrium model systems [3]. Over the past two decades, they have emerged as a kind of ‘laboratory’ in which nonequilibrium phase transitions can be studied. ASEPs have also been invoked to model a variety of physical systems, such as biological transport, traffic problems and gel electrophoresis. Defined on a regular *one-dimensional* lattice of length  $L$ , particles and holes perform nearest-neighbour exchanges biased in a specific direction. If exchanges in the suppressed direction are completely prohibited, the process becomes totally asymmetric (TASEP). For periodic boundary conditions, the TASEP stationary state is

uniform; in contrast, if particles are inserted and extracted at the boundaries with certain rates ( $\alpha$  and  $\beta$ , respectively), three distinct phases are observed: a low-density ( $\alpha < 1/2$ ,  $\alpha < \beta$ ), a high-density ( $\alpha > \beta$ ,  $\beta < 1/2$ ) and a maximal current ( $\alpha > 1/2$ ,  $\beta > 1/2$ ) phase [4–6].

If two species of particles are present, phase transitions can occur even in periodic systems [7, 8]. If the two species (labelled ‘positive’ and ‘negative’) are biased in opposite directions, move into vacant sites (holes) with unit rate and exchange with the opposite species with another rate ( $\gamma < 1$ ), then solid clusters of particles separated by nearly empty domains form easily. Though this particular model is proved to display only a homogeneous phase [9], dramatically different properties emerge when it is generalized in a seemingly trivial manner—from  $1 \times L$  to  $2 \times L$ , modelling a ‘two lane road’ (see [10] and references therein).

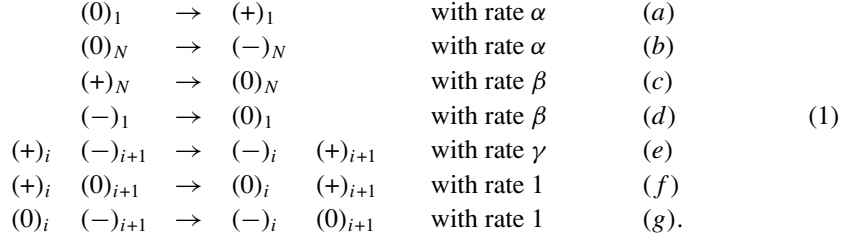
In this letter, we consider the same biased two-species dynamics, but with open boundaries. In this ‘bridge’ model, positive (negative) particles enter the system at the left (right) with rate  $\alpha$  and exit at the right (left) with rate  $\beta$ , reminiscent of traffic crossing a long road bridge in both directions. While cars move freely on clear stretches, they slow down for on-coming traffic, reflected in the bulk mobility  $\gamma$ . This dynamics is clearly symmetric with respect to charge-parity (CP) transformations. The first study [11] of this model focused mostly on  $\gamma = 1$  (i.e., particle–particle and particle–hole exchanges occur with the same rate in the bulk) and reported four phases. Two of these phases are characterized by CP-symmetric steady-state particle densities and currents, while the other two spontaneously break the CP symmetry. Specifically, choosing  $\alpha = 1$  and lowering  $\beta$ , the system undergoes a transition from a symmetric (the so-called power-law) phase with maximal current, equal particle densities and no holes to a low-density symmetric (ld-S) phase where the particle densities are still equal but below  $1/2$ . As  $\beta$  is lowered further, spontaneous symmetry breaking occurs: while both densities remain below  $1/2$ , the two species are associated with different currents and densities. At even lower  $\beta$ , this ‘low-density low-density’ (ld/ld) phase undergoes a final transition into a ‘high-density low-density’ (hd/ld) phase in which one of the particle densities exceeds  $1/2$ . While the existence of the two symmetric phases is supported by an exact solution for  $\beta = 1$ , evidence for the two asymmetric phases is restricted to a mean-field theory and simulation data for fairly small systems. The existence of the (ld/ld)-phase was subsequently disputed [12], based predominantly on simulations of a suitably defined free energy functional which can be extracted from particle density histograms. Arguing that this functional is not very sensitive to the subtle differences between the (ld/ld)- and (hd/ld)-phases, more accurate data for particle density histograms were analysed in [13], providing further evidence for the (ld/ld)-phase albeit with some unusual finite-size properties.

In this letter, we investigate this model further, performing high-precision Monte Carlo (MC) simulations for much larger system sizes and an expanded range of  $\gamma$ , with  $0 < \gamma \leq 1$ , in an attempt to clarify the nature of the (ld/ld)-phase. We present very strong numerical evidence that only a single symmetry breaking transition persists in the large  $L$  or small  $\gamma$  limit, in the sense that the range in  $\beta$  over which the (ld/ld)-phase is observed narrows until it becomes unresolvable. As in the periodic case [10], the behaviour of low-dimensional systems can easily be misextrapolated unless very careful finite-size analyses are performed. Our findings confirm how subtle finite-size effects can be when studying driven diffusive systems. The importance of exploring larger and larger systems cannot be overemphasized. Since a reliable extrapolation of MC data is essential for the analysis of many nonequilibrium phenomena, the understanding of such peculiar finite-size effects may have repercussions well beyond these simple models.

This letter is organized as follows. We first introduce the model and our simulation method, then summarize our data and conclude with a brief summary. More details will be reported in [14].

## 2. Model and methods

The model is defined on a one-dimensional lattice of length  $L$ . An occupation variable  $s_i = (\cdot)_i$  is assigned to each site  $i = 1, \dots, L$ , indicating its state as empty,  $(0)_i$ , or occupied by a positive,  $(+)_i$ , or negative,  $(-)_i$ , particle. Positive particles enter the system on the left with rate  $\alpha$  and move to the right with rate 1 if their neighbour is a hole, or exchange places with a negative particle with rate  $\gamma$ . At the right edge, they leave the system with rate  $\beta$ . Negative particles enter the system from the right and exit at the left, in a CP-invariant fashion. In detail,



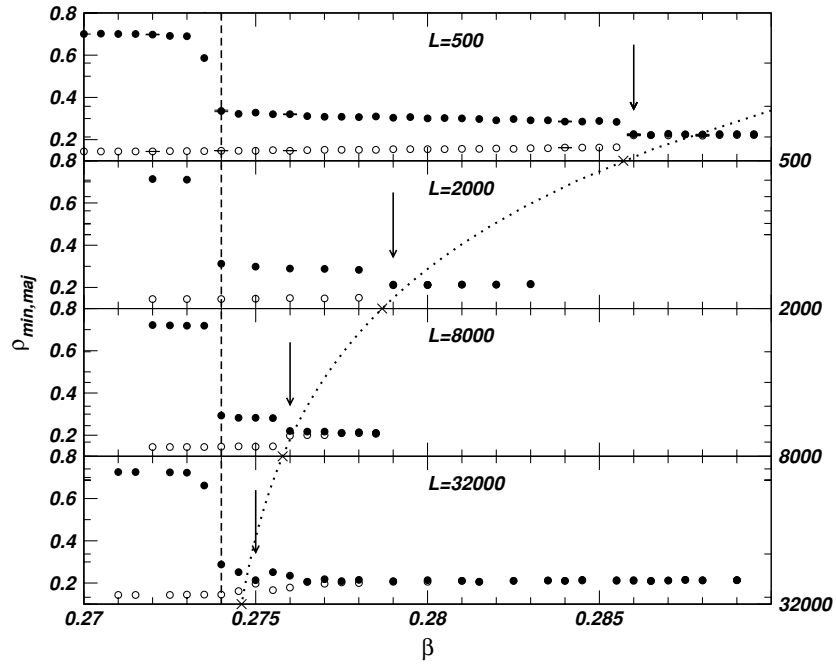
To implement this dynamics, we maintain a list of possible moves, draw randomly from that list and update according to the rates in (1). If the length of the list is  $l$ , continuous time, measured in units of MC steps (MCS), is incremented by  $1/l$  whenever a move is attempted. Taking the average of an observable then requires a weighting by  $1/l$ . The performance of the code can be greatly improved if we note that the particle density changes only at the boundaries, (1)(a)–(d) [15].

All simulations are performed at  $\alpha = 1$ . Starting from an initially randomly half-filled system, we discard at least  $10^9$  MCS to avoid transients, and then gather data for about  $4 \times 10^9$  MCS, unless otherwise noted. Smaller values of  $\gamma$  require longer runs, up to  $2.5 \times 10^{10}$  MCS, to gain a clear picture. The random number generator is the ‘Mersenne twister’ [16].

We generate particle density histograms,  $P_L(\rho_-, \rho_+)$ , by continuously monitoring the instantaneous densities of positive and negative particles,  $\rho_{\pm} \equiv (2L)^{-1} \sum_i [|s_i| \pm s_i]$ , in steady state. The associated configurational averages are denoted by  $\langle \rho_+ \rangle$  and  $\langle \rho_- \rangle$ .

In steady state, the probability of a particular configuration of a *finite* system is rigorously symmetric under CP, so that  $P_L(\rho_-, \rho_+) = P_L(\rho_+, \rho_-)$  and  $\langle \rho_- \rangle = \langle \rho_+ \rangle$ . We exploit this symmetry in order to reduce noise in our statistical data, by considering the ‘folded’ histogram  $P_L(\rho_-, \rho_+) + P_L(\rho_+, \rho_-)$ . In certain sectors of parameter space, CP symmetry is spontaneously broken [11], in the sense that larger systems are ‘trapped’ for times  $\propto \exp(\text{const} \times L)$  in particular regions of phase space with asymmetric particle densities and currents.

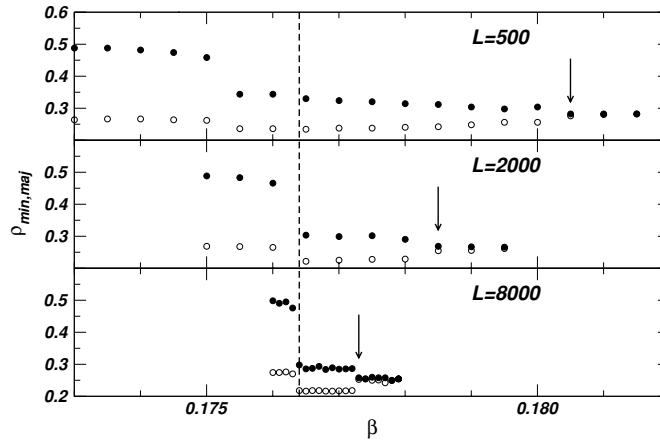
Even for short switching times, the particle density histograms  $P_L(\rho_-, \rho_+)$  carry by far the clearest signature of the transitions [12, 13]. A single peak on the diagonal indicates a symmetric phase, while a double peak with two off-diagonal maxima signals a symmetry-broken phase. The transition between the two asymmetric phases is marked by histograms with two long ridges, one running close and nearly parallel to the  $\rho_-$ -axis, and the other close to the  $\rho_+$ -axis. At the transition, the global maximum suddenly shifts from the near (ld/ld) to the far (hd/ld) end of each ridge, like water sloshing in a narrow channel. Going beyond earlier studies, we *quantify* the histogram features by measuring the observables  $\rho_{\min}$  and  $\rho_{\text{maj}}$ , denoting the density of the minority ( $\rho_{\min}$ ) and majority ( $\rho_{\text{maj}}$ ) species at the *global* maximum of  $P_L(\rho_-, \rho_+)$ . The statistical error of the peak position can be determined from its scatter in independent runs. This is computationally very costly, so that the error bars shown for  $L = 500$  in figure 1 are extrapolated from a few runs at representative parameter values. They are typically of the size of the symbols.



**Figure 1.** The position of the global maximum in the histogram  $P_L(\rho_-, \rho_+)$  as a function of  $\beta$  at  $\gamma = 1$  for different system sizes  $L$ . The filled (open) symbols show the density of the majority (minority) species,  $\rho_{\text{maj}}$  and  $\rho_{\text{min}}$ , respectively. The smallest value of  $\beta$  in the symmetric phase is marked by an arrow, indicating the onset of the (ld-S)–(ld/ld) transition. It moves significantly with  $L$ . In contrast, the (ld/ld)–(hd/ld) transition, marked by a dashed line through all four panels (smallest  $\beta$  in (ld/ld)), remains fixed. The dotted line (right ordinate) is a power-law fit of the position of the (ld-S)–(ld/ld) transition,  $L$  versus  $\beta_2(L)$ , the crosses marking the suggested transition point for each system size shown.

### 3. Results

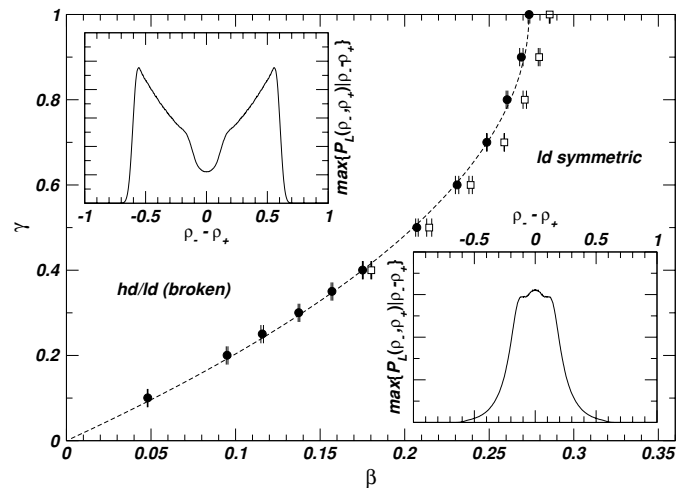
We first consider the  $L$ -dependence of the phase diagram, for  $\gamma = 1$ . Figure 1 shows  $\rho_{\text{min}}$  and  $\rho_{\text{maj}}$  for different system sizes  $L = 500, 2000, 8000, 32\,000$  at  $\gamma = 1$ . Starting at low  $\beta$ , each system size displays two branches, the upper (lower) being associated with  $\rho_{\text{maj}}$  ( $\rho_{\text{min}}$ ). The (hd/ld)–(ld/ld) transition, associated with the abrupt drop in  $\rho_{\text{maj}}$ , occurs at a nearly  $L$ -independent value of  $\beta_1 \simeq 0.2735$ . Even though the difference  $\rho_{\text{maj}} - \rho_{\text{min}}$  is significantly reduced at the transition, it remains nonzero above  $\beta_1$ , indicating *asymmetric* phases on both sides. Remarkably, only  $\rho_{\text{maj}}$  signals the transition clearly while  $\rho_{\text{min}}$  is quite smooth. As we increase  $\beta$  further, we encounter the (ld/ld)–(ld-S) transition where  $\rho_{\text{maj}} - \rho_{\text{min}}$  vanishes, up to statistical fluctuations. Labelling the transition point  $\beta_2(L)$ , it is obvious that  $\beta_2$  depends strongly on  $L$ . Our data for the *smallest* system size (500) confirm the results of [13], i.e.,  $\beta_1 \simeq 0.274$  and  $\beta_2(500) \simeq 0.284$ . However, the rest of our data clearly show that  $\beta_2(L)$  is monotonically *decreasing* with  $L$ . Although the data for  $L = 32\,000$  are quite noisy (with runs of only  $2.2 \times 10^8$  MCS overall), one can still roughly locate the second transition. In particular, we expect  $\beta_2(L)$  to be just below the smallest  $\beta$  for which  $\rho_{\text{maj}} = \rho_{\text{min}}$  (indicated by arrows in figure 1). From these estimates of  $\beta_2(L)$ , we found that the data are remarkably consistent with the power-law  $\beta_2(L) - \beta_1 \propto L^{-0.63}$ . The dotted line in figure 1 shows such a



**Figure 2.** As figure 1, the position of the global maximum in the histogram  $P_L(\rho_-, \rho_+)$  as a function of  $\beta$  for different system sizes  $L$ , but now for  $\gamma = 0.4$ . Filled (open) symbols denote the majority (minority) density,  $\rho_{\text{maj}}$  ( $\rho_{\text{min}}$ ). The position of the transition (ld-S)–(ld/ld), marked by arrows, moves towards the (ld/ld)–(hd/ld) transition. The latter moves slightly towards larger  $\beta$  with  $L$ , the dashed line marking the largest  $\beta$  in (ld/ld) for the largest  $L$ .

fit. Barring further surprises at even larger values of  $L$ , we take this as a strong indication that the (ld/ld)-phase does not persist in the  $L \rightarrow \infty$  limit. As a further test, we investigated the maximum of  $P_L(\rho_-, \rho_+)$ ,  $\max_{\rho_-, \rho_+} \{P_L(\rho_-, \rho_+) | \rho_- - \rho_+\}$ , along diagonals defined by constant  $\rho_- - \rho_+$  [13]. Examples for these ‘ridge profiles’ or ‘silhouettes’ are shown in figure 3. For lower values of  $L$  and fixed  $\beta$  with, say,  $\beta_1 < \beta < \beta_2(500)$ , the ratio of peak to saddle height of this function increases with  $L$ , which appears to support the presence of the (ld/ld)-phase (see figure 10 in [13]). Yet for sufficiently large  $L$ , this ratio decreases again and eventually drops to 1 as soon as  $\beta_2$ , the position of the (ld/ld)–(ld-S) transition, moves below the  $\beta$  considered. We expect to see similar behaviour for any value of  $\beta$  between 0.2735 and 0.284.

These data provide a clear picture of the  $\gamma = 1$  model. We now turn to an expanded parameter space by considering smaller values of  $\gamma$ , reflecting congestions on the bridge. Based on extensive simulations (up to  $5 \times 10^9$  and  $2 \times 10^{10}$  MCS for the transient and statistics, respectively), the transitions can be investigated for  $\gamma$  as low as 0.4. Figure 2 shows, similar to figure 1, the behaviour of minority and majority species. Our key observation is that the width of the (ld/ld)-phase is even further reduced. For  $\gamma = 0.4$  and  $L = 500$ , it is already confined to  $0.175 \leq \beta \leq 0.180$ , shrinking further to  $0.1763 \leq \beta \leq 0.1773$  for  $L = 8000$ , with errors of at most  $\pm 0.0005$ . Unless very finely grained steps in  $\beta$  are simulated, it is easy to miss this phase entirely. It is also apparent that both transitions move to smaller values of  $\beta$  as  $\gamma$  decreases; however,  $\beta_2$  decreases faster so that the width of the (ld/ld)-phase shrinks as  $\gamma$  is lowered. Below  $\gamma = 0.4$  the data become very noisy so that only moderately large systems can be considered. Here, the histograms develop only short ridges, very close to the diagonal  $\rho_+ = \rho_-$ . For small  $\beta$ , the global maximum is at the far end of the ridge, corresponding to the (hd/ld)-phase. As  $\beta$  increases, the ridge shortens and moves closer to the diagonal. During this reshaping, it becomes virtually impossible to resolve whether the global maximum actually moves to the near end of the ridge—which would be the signature of the (ld/ld)-phase—or whether it just immediately appears on the diagonal, signalling the (ld-S)-phase. Also, the low- $\beta$  phase is no longer characterized by having  $\rho_{\text{maj}} > 1/2$ ; instead, both  $\rho_{\text{maj}}$  and  $\rho_{\text{min}}$  are below  $1/2$ ; in this sense, the terminology (hd/ld) is somewhat of a misnomer. To conclude,



**Figure 3.** Phase diagram in  $\beta$ ,  $\gamma$  space for the bridge model at  $L = 500$ . The closed symbols show the transition (hd/ld)–(ld/ld), the open symbols the transition (ld/ld)–(ld-S), which is expected to merge with the former. The error bars indicate the estimated range of the transition. The dashed line is a rough fit of the first transition to  $1 - \sqrt{1 - \beta/\beta_1}$  with  $\beta_1 = 0.2735$ . The two insets show typical ‘ridge profiles’, i.e., the maximum of  $P_L(\rho_-, \rho_+)$  at given  $\rho_- - \rho_+$ , for the two major phases at  $\beta = 0.271$  and  $\beta = 0.287$ , both for  $\gamma = 1$  and  $L = 500$ .

the lowest  $\gamma$  for which we still identify the (ld/ld)-phase with reasonable certainty is  $\gamma = 0.4$  for  $L = 500$ . The phase diagram in figure 3 summarizes our findings, showing the position of the (hd/ld)–(ld/ld) and (ld/ld)–(ld-S) transitions in the  $\beta$ ,  $\gamma$ -plane for  $L = 500$ . Below  $\gamma = 0.4$ , we draw only a single phase boundary, separating a symmetry-broken phase from a symmetric one. Remarkably, the whole length of the (hd/ld)-phase boundary can be well approximated by the phenomenological formula  $1 - \sqrt{1 - \beta/\beta_1}$  with  $\beta_1 = 0.2735$ . If a theoretical framework to predict this formula were developed, it would undoubtedly provide significant insight into the complex behaviour of this simple model.

#### 4. Conclusions

Using extensive simulations for much larger system sizes than previously studied, we performed a finite-size analysis for a simple two-species asymmetric exclusion process with open boundaries. The model is reminiscent of traffic crossing a narrow bridge in both directions. Extending earlier work, we considered the effect of having a small passing probability,  $\gamma$ , which is the rate for two particles to exchange places. In smaller systems with  $\gamma \lesssim 1$ , we confirm the existence of four phases. However, as  $L$  increases or  $\gamma$  decreases, the (ld/ld)-sector shrinks until it can no longer be resolved for  $\gamma \leq 0.4$  and  $L \geq 500$ . This suggests strongly that the (ld/ld)-phase is a finite-size effect and that there is only one symmetry-breaking transition in this model. While mean-field theory predicts otherwise, it is often unreliable in low dimensions. Our investigation highlights again the importance of using very large systems in numerical studies of driven diffusive systems. Moreover, it suggests that scanning a wider range of control parameters may already reveal the same subtleties in smaller systems. Until a reliable analytic approach for this model is found, we believe that our study provides the best insight so far into the issues raised in earlier works [12, 13].

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